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PURE AND ALMOST PURE GOLDEN GRAPHS

Kishori P. Narayankar[1] and D. Shubhalakshmi[2]

Abstract-A $u-v$ path in a graph G is called a peripheral path if there is a path connecting u and v of length equal to the diameter of a graph G. A peripheral path matrix $M_p(G)$ of a graph is a $n \times n$ matrix whose entries p_{ij} are equal to one, if there is a peripheral path between v_i and v_j and zero otherwise. A graph G is said to be pure golden graph if all the non-zero p-eigenvalues of G are $\left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right)$. A graph G is said to be almost pure golden graph if all the non-zero p-eigenvalues are $\left(\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \alpha\right)$ where, $\alpha \neq \frac{-1+\sqrt{5}}{2}$ and $\alpha \neq \frac{-1-\sqrt{5}}{2}$.
In this paper construction of pure golden graph is given. Any n vertex tree T is not a pure golden graph is proved. Also, for any $n \geq 3$, cycle C_n is almost pure golden graph if and only if $n = 5$ is proved.

Keywords: Distance (in Graphs), Peripheral path, Peripheral vertices, Peripheral path matrix of a graph, Eigenvalues, Spectrum.

I. INTRODUCTION

Graphs considered in this paper are finite, simple and unless stated otherwise, also connected. Throughout this paper, we will use the graph-theoretical notation from [1]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ and let $|V(G)| = n$ and $|E(G)| = m$. Let u and v be two vertices of a graph G. The *distance* $d(u,v)$ between the vertices u and v is the length of a shortest path connecting u and v. The *eccentricity* $e(u)$ of the vertex u is $max\{d(u,v) : v \in V(G)\}$. The *radius* $rad(G)$ and *diameter* $diam(G)$ are the minimum and maximum eccentricity, respectively. A vertex u with $e(u) = diam(G)$ is called a peripheral vertex of G. A set of peripheral vertices of G is called as *periphery* and is denoted as $P(G)$. The cartesian product $G \times H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph with vertex set $V(G) \times V(H)$ where vertex (a,x) is adjacent to vertex (b,y) whenever $ab \in E(G)$ and $x = y$ or $a = b$ and $xy \in E(H)$.

The golden ratio (symbol in the Greek latter "phi" φ) is a special number $\varphi = \frac{1+\sqrt{5}}{2}$, approximately equal to 1.6180. Our spectral graph theoretic terminology follows that of the book [3].

Periperal path matrix of G is $n \times n$ matrix,

$$M_p(G) = [p_{ij}] = \begin{cases} 1, & if\ there\ is\ a\ peripheral\ path\ between\ v_i\ and\ v_j \\ 0, & otherwise \end{cases}$$

The characteristic polynomial of $M_p(G)$ is $det\left(\alpha I - M_p(G)\right)$, it is called the characteristic polynomial of G and is denoted by

$$\Psi(G, \alpha) = c_0\alpha^n + c_1\alpha^{n-1} + c_2\alpha^{n-2} + \ldots + c_n.$$

[1] *Department of Mathematics Mangalore University, Mangalore India*
[2] *Department of Mathematics Mangalore University, Mangalore India*

The roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the polynomial Ψ(G,α) are called the eigenvalues of $M_p(G)$. We call the eigenvalues of $M_p(G)$ as the peripheral path eigenvalues (or p-eigenvalues (in short)) of G. Since $M_p(G)$ is a real symmetric matrix, the p-eigenvalues are real and can be ordered in non-increasing order, $\alpha_1 \geq \alpha_2 \geq \ldots$
$\geq \alpha_n$. The p -spectrum of a graph G is the set of eigenvalues of $M_p(G)$ together with their multiplicities. If the distinct p-eigenvalues of $M_p(G)$ are, $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ and their multiplicities are $m(\alpha_1), m(\alpha_2), \ldots, m(\alpha_n)$, then we shall write,

$$p\text{ - spec}(G) = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \mathrm{m}(\alpha_1) & \mathrm{m}(\alpha_2) & \cdots & \mathrm{m}(\alpha_n) \end{pmatrix}$$

***Proposition* 1.1**. [7] Let Ψ (G, α) = $c_0\alpha^n + c_1\alpha^{n-1} + c_2\alpha^{n-2} + \ldots + c_n$ be the characteristic polynomial of
the graph G with respect to peripheral path matrix, then the coefficients of Ψ (G, α) satisfy the following
conditions.

1. $c_0 = 1$
2. $c_1 = 0$
3. $-c_2 = \frac{1}{2}\sum_{i=1}^{n}|\varepsilon_{v_i}|$ = number of peripheral paths of G.
4. $-c_3 = 2\delta$, where δ is the triple of vertices (say v_1, v_2, v_3,) which are at the same distance and $d(v_1, v_2) = d(v_1, v_3) = d(v_2, v_3) = diam(G)$.

***Lemma* 1.2**. [2] Let

$$M_2 = \begin{pmatrix} \mathrm{A}_0 & \mathrm{A}_1 \\ \mathrm{A}_1 & \mathrm{A}_0 \end{pmatrix}$$

be a symmetric 2×2 block matrix. Then the spectrum of M_2 is the union of the spectra of $A_0 + A_1$ and $A_0 - A_1$.

Definition 1.3. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $V_1 \cap V_2 = \emptyset$ be any two graphs. The join, denoted by $G_1 + G_2$ is the graph obtained by joining every vertex of G_1 with every vertex of G_2.

Theorem 1.4. [4] Suppose G is a graph of order n with $k \geq 2$, peripheral vertices such that all these k peripheral vertices are at the same distance from each other in G. Then, G is an integral graph.

Theorem 1.5. [4] Suppose C_n be a cycle on $n \geq 4$ vertices. Then C_n is an integral if and only if n is even.

Corollary 1.6. [4] Odd cycle C_n is integral if and only if n = 3.

Theorem 1.7. [4] Suppose T is a tree of order n with k peripheral vertices
such that $k_1 = k_2 = \ldots = k_l = k$. Then, T is an integral, where $|P(T)| = \sum_{i=1}^{l}\mathrm{P_i(T)}$, such that $d(v_{P_i}(T), v_{P_j}(T)) = diam(T)$ but $d(v_{P_i}(T), v_{P_i}(T)) < diam(T)$, where $\mathrm{P_i(T)}$ is partition of periphery $P(T)$ of T with $|\mathrm{P_i(T)}| = \mathrm{k}_i$, i = 1,2, . . . ,l and $d(v_{P_i}(T), v_{P_j}(T))$ is the distance between the vertex $v \in \mathrm{P_i(T)}$ and $v \in \mathrm{P_j(T)}$.

2 Pure Golden Graph

In analogous to the definition of pure golden graph with respect to adjacency matrix [6] we define, the pure golden graph with respect to peripheral path matrix $M_p(G)$ as follows:

Definition 2.1. A graph G is said to be pure golden graph if all the non-zero p -eigenvalues of G are $\left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right)$.

Example: Let P_4 be a path on 4 vertices and K_1 be a complete graph. Then $P_4 + K_1$, graph has p-eigenvalues $\left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right)$. These are nothing but golden ratios. Hence $P_4 + K_1$ is a pure golden graph.

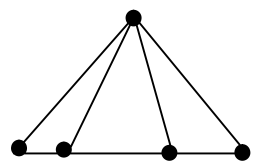
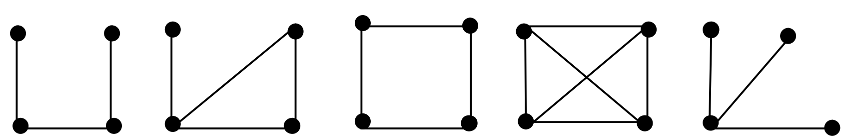


Figure 1: Pure golden graph G of order 5

Proposition 2.2. Graph $P_4 + K_1$ is the smallest pure golden graph.

Proof.: We claim that $P_4 + K_1$is the smallest pure golden graph. If not then there exist a graph G of order less than 5 which is a golden graph. Since pure golden graph has at least 4 non-zero p -eigenvalues, the order of G, is atleast 4. All connected graphs with order 4 are shown in Figure 2.



G_1: G_2: G_3: G_4:

G_5 :

Figure 2: All graphs of order 4.

By calculating the spectrum of G_2 and from **Theorem 1.4** and **Theorem 1.7** , we observe that all these graphs are integral graphs. A contradiction to the fact that G is a golden graph. Hence the claim.

Lemma 2.3. [9] Let G be a graph of order n with k peripheral vertices and K_2 be a complete graph on 2 vertices then,

$$M_p(G \times K_2) = \begin{pmatrix} 0 & M_p(G) \\ M_p(G) & 0 \end{pmatrix}$$

Theorem 2.4. Let $G = P_4 + K_1$ be a pure golden graph and P_l be a path on l vertices. Then, $H = G \times P_l$ is a pure golden graph.

Proof : Denote the vertices of the graph G as $v_1,\ v_2, v_3, v_4,\ v_5$ and vertices of the Path P_l by $u_1 u_2, ,\ \ldots u_l$.
Hence vertices of H are $(v_1\ ,\ u_1), (v_1\ , u2\ ),\ \ldots,\ (v_1, ul\ ), (v_2\ , u_1), (v_2, u2\ ),\ \ldots\ (v_2,\ ul\ ), \ldots, (v_5,\ u_1\ ), (v_5\ , u2), \ldots, (v_5,\ ul\ )$. Clearly
$(v_1, u_1), (v_2,\ u_1\ ), (v_3, u_1), (v_4, u_1), (v_1, ul), (v_2, ul), (v_3, ul), (v_4, ul\ )$ are the 8 peripheral vertices of H. Denote the peripheral vertices of H as $w_{1,} w_{2,}, \ldots, w_8$. Peripheral path matrix of the graph H is,

$$M_p(H) = \left(\begin{array}{cccccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}\right)_{5l \times 5l}$$

$$\cong \left(\begin{array}{c|c} A_{8\times 8} & B_{8\times n-8} \\ {B^T}_{n-8\times 8} & C_{n-8\times n-8} \end{array}\right)$$

where the submatrices B, B^T, C are zero matrices and A is the non-zero submatrix. Thus the non-zero p -eigenvalues of $M_p(H)$ is the non-zero p–eigenvalues of matrix A. Again matrix A can be sub divided into block matrices as,

$$A = \left(\begin{array}{c|c} 0_{4\times 4} & D_{4\times 4} \\ D_{4\times 4} & 0_{4\times 4} \end{array}\right)$$

where submatrix D is a submatrix of submatrix A. Clearly characteristic polynomial of D is

$$\Psi\ (\mathrm{D}, \alpha) = \alpha^4 + \alpha^2 - 1$$
$$0 = (\alpha^2 + \alpha - 1)(\alpha^2 + \alpha - 1)$$

Which implies $(\alpha^2 - \alpha - 1) = 0$ or $(\alpha^2 + \alpha - 1) = 0$

Therefore, $\left(\frac{\mathbf{1+\sqrt{5}}}{\mathbf{2}}, \frac{\mathbf{1-\sqrt{5}}}{\mathbf{2}}, \frac{\mathbf{-1+\sqrt{5}}}{\mathbf{2}}, \frac{\mathbf{-1-\sqrt{5}}}{\mathbf{2}}\right)$ are the non-zero p -eigenvalues of $M_p(D)$

Hence,

$$p\text{ - spec}(D) = \begin{pmatrix} \frac{\mathbf{1+\sqrt{5}}}{\mathbf{2}} & \frac{\mathbf{-1+\sqrt{5}}}{\mathbf{2}} & \frac{\mathbf{1-\sqrt{5}}}{\mathbf{2}} & \frac{\mathbf{-1-\sqrt{5}}}{\mathbf{2}} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

From **Lemma 1.2** we have, p - spec(A) is the union of p -spec(D) and p -spec$(-D)$.
Thus,

$$p\text{ -spec}(A) = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \\ 2 & 2 & 2 & 2 \end{pmatrix}$$

Hence non-zero eigenvalues of $M_p(H)$ contains eigenvalues $\left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right)$.
Hence H is a pure golden graph.

Theorem 2.5. Let G be a pure golden graph of order n with $diam(G) \geq 2$ and K_2 be a complete graph on 2 vertices. Then, there exist a pure golden graph G_i , $i = 1, 2, \ldots$ of order $n\,2^i$ with $iam(G_i) > 2$, $i = 1, 2, 3, \ldots$. where $G_1 = G \times K_2$, and $G_i = G_{i-1} \times K_2$, $i = 2, 3, \ldots$

Proof: Let G be a pure golden graph of order $n \geq 5$ and K_2 be a complete graph on 2 vertices. From **Lemma 2.3** we have,

$$M_p(G_1) = M_p\,(G \times K_2, \,) = \begin{pmatrix} 0 & M_p(G) \\ M_p(G) & 0 \end{pmatrix}$$

Since G is a pure golden graph of order n its non-zero p -eigenvalues are $\left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right)$
Also from **Lemma 1.2** we have, p -spec($M_p(G_1)$)is the union of p –spec $(M_p(G))$ and p –spec (- $M_p(G)$) Hence p –spec $M_p(G_1)$ contains $\left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right)$ as its p -eigenvalues. So G_1 is a golden graph of order $2n$ as, $G_1 = G \times K_2$. Clearly G_1 contains two copies of G. Hence $O(G) = O(G \times K_2) = 2n$. Similarly $G_2 = G_1 \times K_2$, is a golden graph of order 2^2 n as, $G_2 = G_1 \times K_2$, Clearly G_2 contains two copies of G_1 . Hence $O(G_2) = O(G_1 \times K_2) = 2^2\,\mathrm{n}$. Inductively $G_i = G_{i-1} \times K_2$, is a golden graph of order $2^i\,\mathrm{n}$ as, $G_i = G_{i-1} \times K_2$. Clearly G_i contains two copies of G_{i-1} . Hence $O(G_i) = O(G_{i-1} \times K_2) = 2^i\,\mathrm{n}$, $i \geq 1$.

Corollary 2.6.Let $G = P_4 + K_1$be a pure golden graph of order 5 andK_2 be a complete graph on 2 vertices. Then, there exist a pure golden graphG_i , $i = 1, 2, \ldots$ of order $2^i\,5$with $diam(G_i) > 2$, where $G_1 = G \times K_2$. And$G_i = G_{i-1} \times K_2$, $i = 2, 3, \ldots$

Proof: Proof follows from **Theorem 2.5** by taking $G = P_4 + K_1$.

Theorem 2.7.Any n-vertex tree$T$ is not a pure golden graph.

Proof:Assume that tree T of order n is a pure golden graph. Hence its non-zero p-eigenvalues are $\alpha_1 = \frac{1+\sqrt{5}}{2}$, $\alpha_2 = \frac{1-\sqrt{5}}{2}$, $\alpha_3 = \frac{-1+\sqrt{5}}{2}$, $\alpha_4 = \frac{-1-\sqrt{5}}{2}$.Clearly $\alpha_1 = -\alpha_4$and $\alpha_2 = -\alpha_3$. Let the multiplicities of α_1 and $\alpha_4$be l, multiplicities of α_2 and α_3 be r and assume that zero p - eigenvalues has multiplicity$s$.
Thus the characteristic polynomial of T can be expressed as,

$$\Psi\,(T,\alpha) = (\alpha^2-\alpha_1^2)^l(\alpha^2-\alpha_2^2)^r\,(\alpha-0)^s$$

By expanding the above equation and collecting like powers of α we have,

$$= \alpha^{2l+2r+s} - \left[\binom{l}{1}\alpha_1^2 + \binom{r}{1}\alpha_2^2 + \cdots\right]$$

Clearly$n = 2l + 2r + s$. By **Propositon 1.1**, we have ,

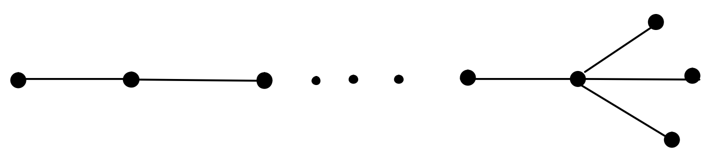
$\binom{l}{1}\alpha_1^2 + \binom{r}{1}\alpha_2^2$ = number of peripheral paths in T.

But $\alpha_1^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$and $\alpha_2^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$

Thus number of peripheral paths in $T = l\left(\frac{3+\sqrt{5}}{2}\right) + \mathrm{r}\left(\frac{3-\sqrt{5}}{2}\right)$

$= \frac{3l+3r}{2} + \frac{\sqrt{5}(l-r)}{2}$.

This equation holds when $l - r = 0$. Since $l > 0$ and $r > 0$, $l = r$must hold.Hence number of peripheral paths in $T = \frac{6l}{2} = 3l$.Since there are 4 non-zero p–eigenvaluestogether with the multiplicity l, numberof peripheral vertices is atleast $4l$. If$l = 1$, then tree T has atleast 4 peripheralvertices with 3 peripheral paths between them and such type of tree T is of theform, as seen in Figure 3.

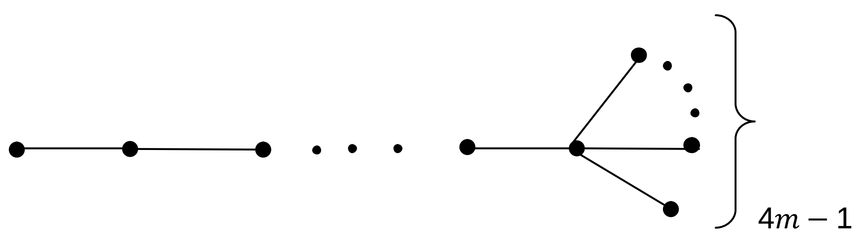


T: Figure 3: Tree T with four peripheral vertices.

But from **Theorem 1.7**, T is an integral tree. A contradiction to the fact that tree T is a pure golden tree and its only non-zero p -eigenvalues are

$$\left(\frac{\mathbf{1}+\sqrt{\mathbf{5}}}{\mathbf{2}},\frac{\mathbf{1}-\sqrt{\mathbf{5}}}{\mathbf{2}},\frac{-\mathbf{1}+\sqrt{\mathbf{5}}}{\mathbf{2}},\frac{-\mathbf{1}-\sqrt{\mathbf{5}}}{\mathbf{2}}\right)$$

If $l = m, m \geq 2$ then tree T has atleast $4m$ peripheral vertices with $3m$ number of peripheral paths between them , which is not possible in a tree because if tree T has $4m$ peripheral vertices then it requires atleast $4m-1$ number of peripheral paths between them., such type of tree T is of the form , as seen in Figure 4.



T:

Figure 4: Tree T with $4m-1$ peripheral vertices.

Hence any n-vertex tree T is not a pure golden graph.

3 Almost Pure Golden Graph.

Definition 3.1. A graph G is said to be almost pure golden graph if all the non-zero p -eigenvalues are $\left(\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \alpha\right)$ where $\alpha \neq \frac{-1+\sqrt{5}}{2}$ and $\alpha \neq \frac{-1-\sqrt{5}}{2}$.

Example: Let C_5 be a cycle on 5 vertices. C_5 has p -eigenvalues $\left(2,\ 2\left(\frac{-1+\sqrt{5}}{2}\right), 2\left(\frac{-1-\sqrt{5}}{2}\right)\right)$

Hence C_5 is an almost pure golden graph.

Proposition 3.2. Cycle on 5 vertices, C_5 is the smallest almost pure golden graph.

Proof: Proof is analogous to the proof of **Proposition 2.2**

Theorem 3.3. For any $n \geq 3$, Cycle C_n is almost pure golden graph if and only if $n = 5$.

Proof: Let C_n be almost pure golden graph. From **Theorem 1.5** and from **Corollary 1.6** we have n is odd and $n \geq 5$.

p -spectrum of C_n is $2\cos\left[\frac{\pi r(n-1)}{n}\right]$, $r = 1, 2, \ldots n$.

Therefore,

$$2\cos\left[\frac{\pi r(n-1)}{n}\right] = \frac{-1-\sqrt{5}}{2}, \quad \text{for some } r.$$

Which implies $\cos\left[\frac{\pi r(n-1)}{n}\right] = \frac{-1-\sqrt{5}}{4}$.

But we known that, $cos 144^0 = \frac{-1-\sqrt{5}}{4}$

Therefore, $\left[\frac{\pi r(n-1)}{n}\right] = 2l\pi \pm 144^0$

Which implies $\frac{n-1}{n} = \frac{10l \pm 4}{5r}$

Here $l = 0$ because l being the number of full rotations and π representing only half of the rotation.

Hence, $n = \frac{5r}{5r-4}$, for this we get integer value only when $r = 1$ and when $r = 1, n = 5$.

Hence cycle C_n is almost pure golden graph when $n = 5$.

Converse part is obvious.

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